Math 407 Example Final Example

1. You will be expected to know the the duality theorems in

   `http://www.math.washington.edu/~burke/crs/407/notes/section4.pdf`

   For example:

   (a) State the Weak Duality Theorem.

   (b) State the Strong Duality Theorem.

   (c) State the Complementary Slackness Theorem.

2. (a) You should be able to solve problems using the primal simplex algorithm; for example, show your work in the solution of the problem

$$\begin{array}{rll} \text{maximize} & c^{\mathbf{T}}x & \text{w.r.t} \quad x \in \mathbf{R}^4_+ \\ \text{subject to} & Ax & \leq b \end{array}$$

   where

$$A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 1 & 1 & 2 & -3 \end{pmatrix}, \; b = \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \; c = \begin{pmatrix} 1 \\ 5 \\ 1 \\ 1 \end{pmatrix}$$

   (Hint, some pivots are easier than others.)

   (b) You should be able to solve problems using the dual simplex algorithm; for example, show your work in the solution of the problem

$$\begin{array}{rll} \text{maximize} & c^{\mathbf{T}}x & \text{w.r.t} \quad x \in \mathbf{R}^4_+ \\ \text{subject to} & Ax & \leq b \end{array}$$

   where

$$A = \begin{pmatrix} 1 & 2 & 3 & -1 \\ 1 & 1 & 2 & -3 \end{pmatrix}, \; b = \begin{pmatrix} -5 \\ -3 \end{pmatrix}, \; c = \begin{pmatrix} -1 \\ -5 \\ -1 \\ -1 \end{pmatrix}$$

3. You should be able to apply the the duality theorems to make certain problems easier to solve. For example, prove that the point $x = (0,\ 4/3,\ 2/3,\ 5/3,\ 0)^{\mathbf{T}}$ is optimal (or prove it is not optimal) for the problem

$$\begin{array}{rll} \text{maximize} & c^{\mathbf{T}}x & \text{w.r.t} \quad x \in \mathbf{R}^5_+ \\ \text{subject to} & Ax & \leq b \end{array}$$

   where

$$A = \begin{pmatrix} 1 & 3 & 5 & -2 & 2 \\ 4 & 2 & -2 & 1 & 1 \\ 2 & 4 & 4 & -2 & 5 \\ 3 & 1 & 2 & -1 & -2 \end{pmatrix}, \; b = \begin{pmatrix} 4 \\ 3 \\ 5 \\ 1 \end{pmatrix}, \; c = \begin{pmatrix} 7 \\ 6 \\ 5 \\ -2 \\ 3 \end{pmatrix}$$

4. You should be able to solve word problems; for example,

   An electronics company has a contract to deliver 20,000 radios within the next four weeks. The client is willing to pay \$20 for each radio delivered by the end of the first week, \$18 for those delivered by the end of the second week, \$16 by the end of the third week, and \$14 by the end of the fourth week. Since each worker can assemble only 50 radios per week, the company cannot meet the order with its present labor force of 40; hence it must hire and train temporary help. Any of the experienced workers can be taken off the assembly line to instruct a class of three trainees; after one week of instruction, each of the trainees can either proceed to the assembly line or instruct additional new classes.

   At present, the company has no other contracts; hence some workers may become idle once the delivery is completed. All of them, whether permanent or temporary, must be kept on the payroll until the end of the fourth week. The weekly wages of a worker, whether assembling, instructing, or being idle, are \$200; the weekly wages of a trainee are \$100. The production costs, excluding the worker's wages, are \$5 per radio.

   The company's aim is to maximize the total net profit. Formulate as an Linear programming problem (not necessarily in the standard from). Use the following notation in your formulation:

e_1, e_2, e_3, e_4	number of experienced workers during the week corresponding to the subscript
a_1, a_2, a_3, a_4	number of assemblers during the week corresponding to the subscript
t_1, t_2, t_3, t_4	number of trainees during the week corresponding to the subscript

5. You should be able to convert a linear programming problem to the corresponding Noes-Clp input file; for example, given the problem

$$\begin{array}{lrl} \text{maximize} & c^{\mathbf{T}}x & \text{w.r.t} \quad x \in \mathbf{R}^2 \times \mathbf{R}^3_+ \\ \text{subject to} & A_{1,1}x_1 + \cdots + A_{1,5}x_5 & = b_1 \\ & A_{2,1}x_1 + \cdots + A_{2,5}x_5 & \geq b_2 \\ & A_{3,1}x_1 + \cdots + A_{3,5}x_5 & \leq b_3 \end{array}$$

   where

$$A = \begin{pmatrix} 1 & 3 & 5 & -2 & 2 \\ 4 & 2 & -2 & 1 & 1 \\ 2 & 4 & 4 & -2 & 5 \end{pmatrix}, \; b = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}, \; c = \begin{pmatrix} 7 \\ 6 \\ 5 \\ -2 \\ 3 \end{pmatrix}$$

   fill in the the characters above the underlines _ (except for the ones that should be left as spaces) so that the corresponding Neos-Clp input solves the problem above:

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ROWS
__  z
__  r1
__  r2
```

```
-- r3
COLUMNS
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    --        --        --             --        --
    --        --        --             --        --
    --        --        --             --        --
    --        --        --             --        --
    --        --        --             --        --
    --        --        --             --        --
    --        --        --             --        --
    --        --        --             --        --
    --        --        --             --        --
RHS
    b         r1        --
    b         r2        --
    b         r3        --
BOUNDS
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    --        --        --
    --        --        --
    --        --        --
    --        --        --
ENDATA
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solution -
]]></param>
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6. Given a Lagrangian you should be able to write down the corresponding primal and dual problems. For example; suppose that $X = \mathbf{R}\times\mathbf{R}_+\times\mathbf{R}_+$, $Y = \mathbf{R}\times\mathbf{R}_+\times\mathbf{R}$, and $L : X\times Y \to \mathbf{R}$ is defined by

$$\begin{aligned} L(x_1, x_2, x_3, y_1, y_2, y_3) &= 3x_1 - 30x_2 + x_3 \\ &+ y_1(x_1 - x_2 + x_3) \\ &+ y_2(5 - x_1 - 5x_3) \\ &+ y_3(-1 + 2x_1 - 5x_2 + x_3) \end{aligned}$$

   (a) Write down the linear programming problem that is equivalent to maximizing $f : X \to \bar{\mathbf{R}}$ defined by

$$f(x) = \inf\{L(x, y) \mid y \in Y\}$$

   with respect to $x \in X$.

   (b) Write down the linear programming problem that is equivalent to minimizing $g : Y \to \bar{\mathbf{R}}$ defined by

$$g(y) = \sup\{L(x, y) \mid x \in X\}$$

   with respect to $y \in Y$.

7. You should understand the properties related to the final simplex tableau. For example, suppose the final tableau of a simple iteration is

x1	x2	x3	s1	s2	s3	b
1	2	0	2	0	-1	2
0	-5	0	-2	1	0	1
0	-1	1	-3	0	2	1
0	-3	0	-1	0	-1	-13

where x_1, x_2, x_3 are the primal variables, s_1, s_2, s_3 are the slack variables, and b_1, b_2, b_3 are the corresponding constraint bounds.

(a) What is the dictionary corresponding to this tableau ?

(b) What is the optimal solution for the primal problem ?

(c) Where is the optimal value of the objective located in the tableau ?

(d) Where is the solution for the dual problem located in the tableau ?

(e) What is the rate of change of the optimal objective value with respect to changes in the value of b_1 , b_2 and b_3 ?

8. You should know how to include the minimum (or maximum) of a set of linear functions as part of your objective in a linear program. This can be used to find a feasible starting point or for the objective in a matrix game. For example; convert the following to a linear programming problem in standard form (a maximize problem with only non-negative variables and inequality constraints):

$$\begin{array}{lll} \text{minimize} & \max\{y_1, y_2\} & \text{w.r.t} \;\; x_1 \in \mathbf{R}_+ \; , \; (x_2, y_1, y_2) \in \mathbf{R}^3 \\ \text{subject to} & 2x_1 + 3x_2 & = y_1 \\ & 3x_1 + 2x_2 & = y_2 \end{array}$$