

Problem 1 This problem uses the definitions in Professor Burke's notes on Duality.

1. Write down the standard form for an arbitrary primal linear programming problem (the form used by the simplex algorithm). Write down the corresponding dual problem.
2. State the Weak Duality Theorem for this pair of primal and dual problems.
3. State the Strong Duality Theorem for this pair of primal and dual problems.

Solution 1 We are given a matrix $A \in \mathbf{R}^{m \times n}$, a vector $b \in \mathbf{R}^m$, and a vector $c \in \mathbf{R}^n$.

1. The corresponding primal problem is

$$\begin{aligned} (P) \quad & \text{maximize} \quad c^{\mathbf{T}}x \quad \text{w.r.t} \quad x \in \mathbf{R}_+^n \\ & \text{subject to} \quad Ax \leq b \end{aligned}$$

The corresponding dual problem is

$$\begin{aligned} (D) \quad & \text{minimize} \quad b^{\mathbf{T}}y \quad \text{w.r.t} \quad y \in \mathbf{R}_+^m \\ & \text{subject to} \quad A^{\mathbf{T}}y \geq c \end{aligned}$$

2. **Weak Duality Theorem** If x is feasible for problem (P) and y is feasible for problem (D) then

$$c^{\mathbf{T}}x \leq y^{\mathbf{T}}Ax \leq y^{\mathbf{T}}b$$

In particular, if (P) is unbounded, then (D) is infeasible, and if (D) is unbounded, then (P) is infeasible.

3. **Strong Duality Theorem** If either (P) or (D) has finite optimal value (the objective supremum for (P) or the objective infimum for (D) is finite), then both (P) and (D) have finite optimal values that are equal. In addition, there are optimal solutions for both (P) and (D) .

Problem 2

$$\begin{aligned} & \text{maximize} && -x_1 - x_2 - x_3 & \text{w.r.t} && x \in \mathbf{R}_+^3 \\ & \text{subject to} && 2x_1 - 2x_2 + x_3 & \leq && -5 \\ & && -x_1 + x_2 - 2x_3 & \leq && -1 \end{aligned}$$

1. Write down the Phase I feasibility problem (in standard form) for this linear program.
2. Write down the initial tableau for this Phase I problem and perform a simplex pivot that results in a tableau corresponding to a basic feasible solution.
3. Write down the tableau for the original linear program as stated in this problem. Note that it is not primal feasible, but it is dual feasible. Perform one pivot of the dual simplex algorithm on this tableau.

Solution 2

1.

$$\begin{aligned} & \text{maximize} && -x_0 & \text{w.r.t} && (x_0, x_1, x_2, x_3)^T \in \mathbf{R}_+^4 \\ & \text{subject to} && -x_0 + 2x_1 - 2x_2 + x_3 & \leq && -5 \\ & && -x_0 - x_1 + x_2 - 2x_3 & \leq && -1 \end{aligned}$$

2. The initial tableau for the corresponding Phase I problem is

x0	x1	x2	x3	s1	s2	b
-1	2	-2	1	1	0	-5
-1	-1	1	-2	0	1	-1
0	-1	-1	-1	0	0	0
-1	0	0	0	0	0	0

The third row contains the original objective which makes it easier to go on to Phase II, but it is not necessary for the Phase I problem. Pivoting on row 1 and column 1 we have

x0	x1	x2	x3	s1	s2	b
1	-2	2	-1	-1	0	5
0	-3	3	-3	-1	1	4
0	-1	-1	-1	0	0	0
0	-2	2	-1	-1	0	5

for which the basic solution is feasible. Note third row corresponds to original objective and fourth row to the Phase I objective, so the corresponding value of b need not be positive for the basic solution to be feasible.

3. The initial tableau corresponding to the linear program in the problem statement is

x1	x2	x3	s1	s2	b
2	-2	1	1	0	-5
-1	1	-2	0	1	-1
-1	-1	-1	0	0	0

Using the dual simplex algorithm, we pivot on row 1 and column 2 to obtain

x1	x2	x3	s1	s2	b
-1	1	-1/2	-1/2	0	5/2
0	0	-3/2	1/2	1	-7/2
-2	0	-3/2	-1/2	0	5/2

We also could pivot on row 2 and column 3 to obtain

x1	x2	x3	s1	s2	b
3/2	-3/2	0	1	1/2	-11/2
1/2	-1/2	1	0	-1/2	1/2
-1/2	-3/2	0	0	-1/2	1/2

Problem 3 The Three Mines Company owns three different mines that produce an ore which, after being crushed, is graded into three classes: high, medium and low-grade. The company has a contract to provide a smelting plant with 20 tons of high-grade, 20 tons of medium-grade, and 20 tons of low-grade ore per week. The three mines have different operating characteristics as detailed below:

Mine	Cost	High	Medium	Low
One	20	6	4	2
Two	18	4	4	4
Three	16	2	4	6

where Cost is in thousands of dollars per day to operate the corresponding mine, High, Medium and Low are the number of tons per day produces by the corresponding mine. The company is not allowed to operate on Saturday or Sunday. Use x_1 , x_2 , and x_3 to denote the decision variables for this problem. State what the meaning of the decision variables and find a matrix $A \in \mathbf{R}^{6 \times 3}$, a vector $b \in \mathbf{R}^6$, and a vector $c \in \mathbf{R}^3$, such that the optimal solution for

$$\begin{aligned} & \text{maximize } c^T x \text{ w.r.t } x \in \mathbf{R}_+^3 \\ & \text{subject to } Ax \leq b \end{aligned}$$

fulfills the contract with the minimal cost.

Solution 3 We use x_1 , x_2 , and x_3 to denote the number number of days per week that the company operates mine number one, two and three respectively.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -4 & -2 \\ -4 & -4 & -4 \\ -2 & -4 & -6 \end{pmatrix}, b = \begin{pmatrix} 5 \\ 5 \\ 5 \\ -20 \\ -20 \\ -20 \end{pmatrix}, c = \begin{pmatrix} -20 \\ -18 \\ -16 \end{pmatrix}$$

Problem 4 Prove or disprove the hypothesis that $x_1 = 0$, $x_2 = 7$ and $x_3 = 0$ is optimal for the problem

$$\begin{aligned} & \text{maximize} && 3x_1 + 2x_2 + 4x_3 && \text{w.r.t.} && x \in \mathbf{R}_+^3 \\ & \text{subject to} && -x_1 - x_2 - 2x_3 && \leq && -4 \\ & && 2x_1 + 3x_3 && \leq && 5 \\ & && 2x_1 + x_2 + 3x_3 && \leq && 7 \end{aligned}$$

Solution 4

1. The vector x is feasible because:

$$\begin{aligned} -x_1 - x_2 - 2x_3 &= -7 < -4 \\ 2x_1 + 3x_3 &= 0 < 5 \\ 2x_1 + x_2 + 3x_3 &= 7 = 7 \end{aligned}$$

2. If x is optimal then there must be a $y \in \mathbf{R}_+^3$ that is optimal for the dual and that satisfies complementary slackness with respect to x ; i.e.

$$\begin{aligned} x_2 \neq 0 &\Rightarrow -y_1 + y_3 = 2 \\ -x_1 - x_2 - 2x_3 = -7 < -4 &\Rightarrow y_1 = 0 \\ 2x_1 + 3x_3 = 0 < 5 &\Rightarrow y_2 = 0 \end{aligned}$$

Thus, if x is optimal, then the solution of the dual must be $y_1 = 0$, $y_2 = 0$, $y_3 = 2$.

3. The vector y is dual feasible because:

$$\begin{aligned} -y_1 + 2y_2 + 2y_3 &= 4 \geq 3 \\ -y_1 + y_3 &= 2 \geq 2 \\ -2y_1 + 3y_2 + 3y_3 &= 6 \geq 4 \end{aligned}$$

In summary, x is feasible for the primal, y is feasible for the dual, and the pair satisfies the complementary slackness condition. Thus, by the complementary slackness theorem, x is optimal for the primal and y is optimal for the dual.

4. As a check, we note that the primal and dual objectives are equal because

$$\begin{aligned} 3x_1 + 2x_2 + 4x_3 &= 14 \\ -4y_1 + 5y_2 + 7y_3 &= 14 \end{aligned}$$

5. As an alternative solution. One could apply the two phase simplex method to solve the optimization problem. As it turns

Problem 5 Write the dual of the following linear program directly (without converting to the standard form used by the simplex algorithm).

$$\begin{array}{ll} \text{maximize} & x_1 \quad \text{w.r.t.} \quad x_1 \in \mathbf{R}, (x_2, x_3)^{\mathbf{T}} \in \mathbf{R}_+^2 \\ \text{subject to} & x_2 + x_3 = 1 \\ & x_1 - 5x_2 + 10x_3 \leq 0 \\ & x_1 + 5x_2 - 10x_3 \leq 0 \end{array}$$

Solution 5

$$\begin{array}{ll} \text{minimize} & y_1 \quad \text{w.r.t.} \quad y_1 \in \mathbf{R}, (y_2, y_3)^{\mathbf{T}} \in \mathbf{R}_+^2 \\ \text{subject to} & y_2 + y_3 = 1 \\ & y_1 - 5y_2 + 5y_3 \geq 0 \\ & y_1 + 10y_2 - 10y_3 \geq 0 \end{array}$$